

Point particles in 2+1 dimensions: toward a semiclassical loop gravity formulation

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Abstract

We study point particles in 2+1 dimensional first order gravity using a triangulation to fix the connection and frame-field. The Hamiltonian is reduced to a boundary term which yields the total mass. The triangulation is dynamical with non-trivial transitions occurring when a particle meets an edge. This framework facilitates a description in terms of the loop gravity phase space.

1 Introduction

Three dimensional gravity has often been used as a toy model for the four dimensional theory. Here we study point particles in 2+1 dimensions using spatial geometries analogous to those in [1]. These geometries are isomorphic to a gauge-reduced holonomy-flux phase space which descends from the $\hbar \rightarrow 0$ limit of a loop quantum gravity Hilbert space.

We consider a Hamiltonian written in terms of frame-field and connection variables. The spacetime signature is taken to be Euclidean since the relevant gauge group is then $SU(2)$, as in 3+1 dimensions with Lorentzian signature. Our first step is to specify the variables over an entire spacelike slice using a triangulation where particles sit on the vertices. After gauge fixing, the Hamiltonian is reduced to a boundary term which equals the total particle mass. In this gauge the triangulation evolves according to particle momenta and undergoes discrete changes when a par-

ticle meets an edge, collapsing one of the triangles.

2 Hamiltonian formulation

Spacetime is taken to be $M = \mathbb{R} \times \Sigma$ where Σ is a spacelike surface homeomorphic to a disc. The first order formalism of general relativity parameterizes the gravitational field in terms of a connection \mathbf{A} and a frame-field \mathbf{e} , both being one-forms on Σ taking values in the $\mathfrak{su}(2)$ algebra. We use $\mathfrak{su}(2)$ basis elements τ^i (for $i = 0, 1, 2$) which are given by $-i/2$ times the Pauli matrices. Our notation is such that elements of $\mathfrak{su}(2)$ are written as $\mathbf{A} \equiv A^i \tau^i$, for example, and all internal indices are written as superscripts.

In units where $8\pi G = c = 1$ a Hamiltonian for pure gravity is given by [2]:

$$\begin{aligned} H &= - \int_{\Sigma} (N^i F^i + \lambda^i G^i) + \int_{\partial\Sigma} N^i A^i, \\ F^i &= dA^i + \frac{1}{2} \epsilon^{ijk} A^j \wedge A^k, \\ G^i &= de^i + \epsilon^{ijk} A^j \wedge e^k. \end{aligned} \quad (1)$$

N^i and λ^i are Lagrange multipliers for the flatness and Gauss constraints respectively. We normalize the time coordinate by choosing $|N|^2 = 1$. The boundary term and the condition $\lambda^i|_{\partial\Sigma} = 0$ ensure that the variational principal is well-defined and the constraints are first class. The Poisson algebra is:

$$\{A_a^i(x), e_b^j(y)\} = \epsilon_{ab} \delta^{ij} \delta^2(x - y), \quad (2)$$

where $a = 1, 2$ labels the space coordinates.

It is well-known that in 2+1 dimensions point particles represent conical singularities [3]. We introduce particles by replacing the flatness constraint with $F^i - \sum_I \delta^2(x - q_\pi) p_\pi^i dx^2$ so that each particle π gives a contribution to the curvature proportional to its momentum \mathbf{p}_π at the location $x = q_\pi$ [2]. In this treatment the particles are defined by the fields (\mathbf{A}, \mathbf{e}) and do not have independent parameters of their own. We consider the total mass to be less than 2π so that Σ is open [3].

3 Gauge fixing

In order to solve the constraints and reduce the Hamiltonian, we triangulate Σ according to the particle locations so that we may give a piecewise definition of (\mathbf{A}, \mathbf{e}) on $\Sigma \setminus \{q_\pi\}$. See [4] for other approaches along these lines. For clarity, we take the boundary $\partial\Sigma$ to be triangular; the generalization to arbitrary polygons follows simply. We take each particle position to be an internal vertex and connect all vertices with straight edges to obtain a triangulation. There is ambiguity in this procedure, but the number of triangles T is fixed by the number of particles Π to be $T = 2(\Pi + 1) - 1$ via the Euler characteristic for planar graphs. Different triangulations generally lead to different gauge choices.

Within each triangle Δ , we must choose $(\mathbf{A}_\Delta, \mathbf{e}_\Delta)$ such that $G^i = F^i = 0$. The general solution [1] is given by an $SU(2)$ group element a_Δ and a closed one-form χ_Δ :

$$\mathbf{A}_\Delta = a_\Delta da_\Delta^{-1}, \quad \mathbf{e}_\Delta = a_\Delta \chi_\Delta a_\Delta^{-1}. \quad (3)$$

Consider two adjacent triangles labeled 1 and 2. We ensure that the fields are continuous across the shared edge e by requiring that there exists an $h_e \in SU(2)$ on the edge such that:

$$a_2|_e = a_1 h_e|_e, \quad \chi_2|_e = h_e^{-1} \chi_1 h_e|_e. \quad (4)$$

Now, if we integrate the curvature over a region containing a single vertex π we must obtain $\int F^i = p_\pi^i$. In order to specify the group elements a_Δ we subdivide each triangle into three regions defined by

edges joining the centroid to the vertices. Consider one such region Δ_r where points $(0, \pm y_0)$ are the endpoints of an edge, and (x_0, y_1) is the centroid of the triangle. We define:

$$u = \tanh^{-1}(v_+) + \tanh^{-1}(v_-), \quad (5)$$

where $v_\pm = \frac{x}{x_0} \pm \left(\frac{y}{y_0} - \frac{xy_1}{x_0 y_0} \right)$. Note that the line $u = 0$ coincides with the edge, and $u = \infty$ consists of lines between the centroid of the triangle and the endpoints of the edge. We set the group element in a region equal to:

$$a_{\Delta_r}(u) = \exp \left(\mathbf{P} \int_0^u f(\tilde{u}) d\tilde{u} \right), \quad (6)$$

where we have introduced a bump function $f(u)$ which satisfies $\int_0^\infty f(u) du = 1$. For a given u , this group element defines a rotation about a direction $\mathbf{P}/|P|$ by an angle proportional to $|P|$.

With these ingredients it is now possible to choose a gauge, i.e. to specify the frame-field and connection on $\Sigma \setminus \{q_\pi\}$. Given a set of particles $\{q_\pi, p_\pi\}$, one first defines a boundary and introduces a triangulation. Then we arbitrarily choose a triangle (labeled 1) and make a choice for (a_1, χ_1) . Moving to a neighbouring triangle, we specify a_2 here and use (4) to find χ_2 . We continue in this manner until (a_Δ, χ_Δ) (and thereby $(\mathbf{A}_\Delta, \mathbf{e}_\Delta)$) are defined in each triangle, taking care to ensure the connection yields proper curvature at each vertex by choosing the parameter \mathbf{P} in each a_{Δ_r} to be consistent with particle momenta. The aforementioned relationship between the number of triangles and the number of particles ensures that this procedure will work for arbitrarily many particles. See [5] for a more detailed description of this process.

Preserving the gauge dynamically leads to a condition on the Lagrange multipliers:

$$d_A N^i + \epsilon^{ijk} e^j \lambda^k = 0. \quad (7)$$

This fixes three of the six degrees of freedom in N^i and λ^i . We eliminate the remaining ambiguity by choosing $d_A N^i = 0$ and setting $\mathbf{N}(q_\pi) = \mathbf{p}_\pi/m$. This choice ensures that particles move along with the triangulation and remain at the vertices.

4 Dynamics and observables

After specifying (\mathbf{A}, \mathbf{e}) , the Hamiltonian is reduced to a boundary term which evaluates to:

$$\begin{aligned} H &= \int_{\partial\Sigma} N^i A^i = \int_{\Sigma} N^i d_A A^i \\ &= \sum_{\pi} N^i(q_{\pi}) p_{\pi}^i = \sum_{\pi} m_{\pi}, \end{aligned} \quad (8)$$

where we used Stokes' theorem, the conditions on \mathbf{N} , and the flatness constraint.

Dynamics are implicitly determined by the direction of momentum $N^i(q_{\pi})$ for each particle via the equation of motion:

$$\dot{q}_{\pi}^i(t) = \dot{q}_{\pi}^i(0) + t N^i(q_{\pi}), \quad (9)$$

and the triangulation evolves accordingly. Non-trivial, discrete transitions occur when a particle meets an edge. Consider triangles 1 and 2 depicted in fig. 1. If the particle at the top of the left hand

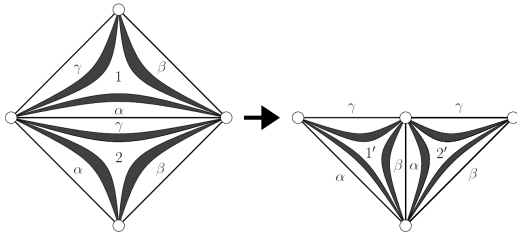


Figure 1: Before and after a discrete change in triangulation. α, β, γ label the three regions of each triangle. Grey regions (bounded by constant- u lines) indicate where the connection is generally non-zero.

side of the figure moves downward until it reaches the shared edge, then 1 closes and 2 splits into 1' and 2' (shown on the right hand side) preserving the total number of triangles. The fields are defined in the new triangles by specifying the SU(2) rotation parameters:

$$\begin{aligned} \mathbf{P}_{1'_{\gamma}} &= \mathbf{P}_{1_{\alpha}} + \mathbf{P}_{2_{\gamma}} + \mathbf{P}_{1_{\gamma}}, & \mathbf{P}_{1'_{\alpha}} &= \mathbf{P}_{2_{\alpha}}, \\ \mathbf{P}_{2'_{\gamma}} &= \mathbf{P}_{1_{\alpha}} + \mathbf{P}_{2_{\gamma}} + \mathbf{P}_{1_{\beta}}, & \mathbf{P}_{2'_{\beta}} &= \mathbf{P}_{2_{\beta}}, \\ \mathbf{P}_{1'_{\beta}} &= \mathbf{P}_{2'_{\alpha}} = 0. \end{aligned} \quad (10)$$

Notice $h_e = 1$ along the new edge shared by 1' and 2' so that $\chi_{1'} = \chi_{2'} = \chi_2$. The fields in all other triangles are unaffected by this transition.

5 Conclusion

We have described a system of point particles in 2+1 dimensional gravity in terms of evolving triangulations. A gauge choice for (\mathbf{A}, \mathbf{e}) was specified by choosing a_{Δ} and χ_{Δ} in each triangle. The triangulation evolves according to particle dynamics so that particles remain at vertices at all times. The discrete change in triangulation that occurs when a vertex meets an edge is well-defined.

This construction yields spatial geometries that are the two-dimensional analog of those in [1]. Having $(a_{\Delta}, \chi_{\Delta})$ in each triangle allows us to immediately write this data in terms of holonomy-flux variables. Moreover, knowing how $(a_{\Delta}, \chi_{\Delta})$ behave under a discrete change in triangulation tells us how the holonomies and fluxes will change. This sets the stage for a semiclassical loop gravity description of the system [5].

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